CU-TP-1101
RUNHETC-2003-31

A First-Quantized Formalism for Cosmological Particle Production

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We show that the amount of particle production in an arbitrary cosmological background can be determined using only the late-time positive-frequency modes. We don't refer to modes at early times, so there is no need for a Bogolubov transformation. We also show that particle production can be extracted from the Feynman propagator in an auxiliary spacetime. This provides a first-quantized formalism for computing particle production which, unlike conventional Bogolubov transformations, may be amenable to a string-theoretic generalization.

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1 Introduction

The standard approach to cosmological particle production [1, 2] involves finding two sets of solutions to the wave equation, with pair production given by a Bogolubov transformation between the two sets of modes. Although this method has a long and proven record of success, there are reasons to search for alternative techniques. In particular, the second-quantized formalism underlying the Bogolubov transformation does not carry over to string theory, whose standard perturbative formulation is first-quantized [3, 4]. In order to estimate, say, the spectrum of density fluctuations coming from stringy effects [5, 6, 7, 8, 9, 10], one would like to have a first-quantized technique in which one can systematically compute α' corrections [11].

In this paper we present a straightforward method for computing pair-production in an arbitrary cosmological background without using Bogolubov transformations. We develop the method in two steps. First we show how to extract pair production from the Feynman propagator using only the late-time positive-frequency modes. Then we show that particle production can

be derived from the Feynman propagator of an auxiliary spacetime, without knowledge of the late-time modes. Thus we can compute particle production in a completely first-quantized framework, without any reference to mode solutions. Similar first-quantized techniques have been successful in calculating pair creation from horizons [12, 13], where again Bogolubov transformations are usually used.

This paper is organized as follows. In section 2 we briefly review the standard treatment of a scalar field in an FRW universe. In section 3 we show that pair production can be obtained from the Feynman propagator using only the late time positive-frequency modes. The number of particles with momentum k at late times, $\langle N_k \rangle$, turns out to be given by a simple expression:

$$\langle N_k \rangle = \frac{|\gamma_k|^2}{1 - |\gamma_k|^2} \qquad \text{where} \qquad |\gamma_k|^2 = \left| \frac{\omega_k - \bar{\omega}_k^*}{\omega_k + \bar{\omega}_k} \right|^2 . \tag{1}$$

Here ω_k is a free parameter that specifies the initial state of this momentum mode of the field. The quantity $\bar{\omega}_k$ is computed from out modes, $u_k(\eta)$, that are positive-frequency at late times:

$$\bar{\omega}_k = i \partial_\eta \log u_k |_{\eta = \eta_0} , \tag{2}$$

where η_0 is the conformal time at which initial conditions are specified. In practice this technique seems more computationally efficient than the conventional approach; in the appendix, we demonstrate the method by calculating particle creation in various cosmological backgrounds. Although the approach so far is perfectly adequate for computing pair creation, it still requires knowing the out modes and consequently does not provide a fully first-quantized formalism. Therefore, in section 4, we show that pair production can be determined from the Feynman propagator in an auxiliary spacetime. This provides a method for calculating $\bar{\omega}_k$ using only first-quantized techniques.

2 Scalar Field Quantization

Consider a free minimally-coupled real scalar field $\hat{\phi}(x)$ in a Robertson-Walker universe with metric

$$ds^2 = a^2(\eta) \left(-d\eta^2 + h_{ij} dx^i dx^j \right) .$$

Here $a(\eta)$ is the scale factor, η is conformal time, and h_{ij} is the metric of a maximally-symmetric n-dimensional space. We make the field redefinition

$$\hat{\phi}(x) = a^{(1-n)/2}(\eta)\hat{\chi}(x) \ , \tag{3}$$

and write $\hat{\chi}(x)$ as a sum over momentum modes:

$$\hat{\chi}(x) = \sum_k \left(\hat{a}_k f_k(\vec{x}) u_k(\eta) + \hat{a}_k^\dagger f_k^*(\vec{x}) u_k^*(\eta) \right) \ . \tag{4}$$

The corresponding mode functions $u_k(\eta)$ satisfy

$$u_k'' + V_k u_k = 0 \tag{5}$$

where

$$V_k(\eta) = k^2 + a^2 m^2 - \frac{1}{2}(n-1)\frac{a''}{a} - \frac{1}{4}(n-1)(n-3)\left(\frac{a'}{a}\right)^2 \ . \tag{6}$$

Here $'$ indicates a derivative with respect to η. If one thinks of η as position, then this is a one-dimensional Schrödinger equation with potential $-\frac{1}{2}V_k(\eta)$.

2.1 Final State

In a general time-dependent background there is no useful sense in which the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ annihilate and create particles. However throughout this paper we will assume that the universe is asymptotically adiabatic in the future. Then the WKB approximation becomes valid as $\eta \to \infty$, and we can take the modes to satisfy [1]

$$u_k(\eta) \sim e^{-i\int^\eta d\eta' \sqrt{V(\eta')}} \quad \text{as } \eta \to +\infty. \tag{7}$$

This means the $u_k(\eta)$ are positive-frequency in the future, while the $u_k^*(\eta)$ are negative-frequency, so the $\hat{a}_k$ are operators which annihilate the preferred future, or out, vacuum:

$$\hat{a}_k |\text{out}\rangle = 0 \qquad \forall k \ . \tag{8}$$

2.2 Initial State

We also need to specify the initial state of the field. We will consider initial states that can be characterized as follows. First choose a conformal time

η_0 at which to fix initial conditions. Then for each momentum k choose a complex parameter ω_k. The ω_k are arbitrary, aside from the fact that for reasons given below we require $\mathrm{Re}\,\omega_k > 0$ and $\omega_k = \omega_{-k}$. Now let $\hat{\chi}_k(\eta)$ be a Fourier component of the field,

$$\hat{\chi}_k(\eta) = \int dx\, f_k^*(\vec{x})\hat{\chi}(\eta,\vec{x}) = \hat{a}_k u_k(\eta) + \hat{a}^\dagger_{-k} u^*_{-k}(\eta)\ . \tag{9}$$

To specify the initial state of the field we define the operators

$$\hat{b}_k \equiv \sqrt{\frac{\omega_k\,|\omega_k|}{2\,\mathrm{Re}\,\omega_k}}\left(\hat{\chi}_k(\eta_0) + \frac{i\hat{\pi}_k(\eta_0)}{\omega_k}\right)\ . \tag{10}$$

Here the conjugate momentum $\hat{\pi}_k(\eta) = \partial_\eta \hat{\chi}_k$ satisfies $i[\hat{\pi}_k, \hat{\chi}_{k'}] = \delta_{k+k'}$ as well as $\hat{\chi}_k^\dagger = \hat{\chi}_{-k}$, $\hat{\pi}_k^\dagger = \hat{\pi}_{-k}$. The normalization of $\hat{b}_k$ and the condition $\omega_k = \omega_{-k}$ ensure that $[\hat{b}_k, \hat{b}^\dagger_{k'}] = \delta_{kk'}$.

We take the initial state to satisfy $\hat{b}_k|\mathrm{in}\rangle = 0$, or equivalently

$$\partial_\eta \hat{\chi}_k(\eta_0)|\mathrm{in}\rangle = i\omega_k \hat{\chi}_k(\eta_0)|\mathrm{in}\rangle\ . \tag{11}$$

The space of initial states we can consider is quite large: the values of the parameters ω_k are essentially arbitrary, so we can specify one complex parameter per momentum mode. By varying the ω_k we can sweep out the entire space of states that are related by Bogolubov transformations.[1]

(Another way to characterize the initial state is to note that, as a consequence of (11), the initial wavefunctional for the field is Gaussian:

$$\Psi[\chi]|_{\eta=\eta_0} \sim \exp -\frac{1}{2}\sum_k \omega_k |\chi_k|^2\ . \tag{12}$$

We require $\mathrm{Re}\,\omega_k > 0$ so that the wavefunctional is normalizable, and $\omega_k = \omega_{-k}$ so that the integrability condition $[\hat{b}_k, \hat{b}_{k'}]|\mathrm{in}\rangle = 0$ can be satisfied.)

2.3 Bogolubov transformations

We now recall some standard results [1, 2]. Bogolubov transformations express the operators $\hat{a}_k$, which annihilate the out-vacuum, in terms of $\hat{b}_k$, and

[1] Although two complex parameters α_k and β_k appear in the Bogolubov transformations (13), they are constrained to satisfy $|\alpha_k|^2 - |\beta_k|^2 = 1$. Also a Bogolubov transformation which merely multiplies $\hat{b}_k$ by a phase is redundant since it leaves $|\mathrm{in}\rangle$ invariant.

vice versa:

$$\hat{a}_k = \sum_{k'} \left(\alpha_{kk'} \hat{b}_{k'} + \beta^*_{-kk'} \hat{b}^\dagger_{k'} \right) \ , \quad \hat{b}_k = \sum_{k'} \left(\alpha^*_{kk'} \hat{a}_{k'} - \beta^*_{-kk'} \hat{a}^\dagger_{k'} \right) \ . \tag{13}$$

The Bogolubov coefficients $\alpha_{kk'}$ and $\beta_{kk'}$ are determined by the overlap of the positive-frequency out-modes with the positive or negative frequency in-modes. The spatial modes are orthonormal, so the Bogolubov coefficients are diagonal in k:

$$\alpha_{kk'} \equiv \alpha_k \delta_{kk'} \qquad \beta_{kk'} \equiv \beta_k \delta_{kk'} \ . \tag{14}$$

The initial state can be written as a squeezed state [14, 15],

$$|\text{in}\rangle = \prod_k C_k \exp\left(-\frac{1}{2} \gamma_k \hat{a}^\dagger_k \hat{a}^\dagger_{-k} \right) |\text{out}\rangle \ , \tag{15}$$

where $C_k = (1 - |\gamma_k|^2)^{1/4}$ so that $\langle\text{in}|\text{in}\rangle = 1$. Requiring that $|\text{in}\rangle$ be annihilated by the $\hat{b}_k$ implies (using (13) and the commutation algebra) that $\gamma_k = -\beta_k/\alpha^*_k$. Note that, by (15),

$$\gamma_k = -\frac{\langle\text{out}|\hat{a}_{-k}\hat{a}_k|\text{in}\rangle}{\langle\text{out}|\text{in}\rangle} \ . \tag{16}$$

We see that γ_k is related to pair creation; indeed, it is the normalized probability amplitude for the in-vacuum to evolve at late times into a state that has two particles in it, $\hat{a}^\dagger_k \hat{a}^\dagger_{-k}|\text{out}\rangle$. The average number of particles produced is given by the expectation value of the number operator,

$$\langle\text{in}|\hat{N}_k|\text{in}\rangle = |\beta_k|^2 = \frac{|\gamma_k|^2}{1 - |\gamma_k|^2} \ , \tag{17}$$

where we have used the normalization condition $|\alpha_k|^2 - |\beta_k|^2 = 1$.

3 Pair Creation Using Only Out-Modes

The Feynman propagator is defined as the vacuum expectation value of the time-ordered product of field operators. However, the vacua appearing on the left and right of the operator product are different:

$$iG_F(k, \eta_1; k', \eta_2) = \frac{\langle\text{out}|T\hat{\chi}_k(\eta_1)\hat{\chi}_{-k'}(\eta_2)|\text{in}\rangle}{\langle\text{out}|\text{in}\rangle} \ . \tag{18}$$

Writing $\hat{\chi}_k$ in terms of modes that are positive-frequency at late times, $\hat{\chi}_k = \hat{a}_k u_k(\eta) + \hat{a}^\dagger_{-k} u^*_{-k}(\eta)$, we find that (for $\eta_1 > \eta_2$)

$$\begin{aligned} iG_F(k,\eta_1;k',\eta_2) &= \frac{\langle \mathrm{out}|\hat{a}_k\hat{a}^\dagger_{k'}|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} u_k(\eta_1)u^*_{k'}(\eta_2) + \frac{\langle \mathrm{out}|\hat{a}_k\hat{a}_{-k'}|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} u_k(\eta_1)u_{-k'}(\eta_2) \\ &= \left(u_k(\eta_1)u^*_k(\eta_2) - \gamma_k u_k(\eta_1)u_{-k}(\eta_2)\right)\delta_{kk'} \ . \end{aligned} \tag{19}$$

To obtain the second term in the second line we used (15) and (16). Since, by (5), $u_k(\eta) = u_{-k}(\eta)$ we will suppress all the k indices henceforth. Then in general the Feynman propagator, expressed in terms of the out-modes, is simply

$$iG_F(\eta_1,\eta_2) = u(\eta_1)u^*(\eta_2)\theta(\eta_{12}) + u(\eta_2)u^*(\eta_1)\theta(\eta_{21}) - \gamma u(\eta_1)u(\eta_2) \ . \tag{20}$$

The first two terms are standard but the third term is a consequence of the in and out vacua being different. Moreover we see that the extra term, being proportional to γ, signals pair creation.

We set the boundary conditions at some time in the far past, η_0, using (11):

$$\partial_{\eta_2} iG_F(\eta_1,\eta_2)\big|_{\eta_2=\eta_0} = \frac{\langle \mathrm{out}|\hat{\chi}(\eta_1)(\partial_\eta\hat{\chi})(\eta_0)|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} = \frac{\langle \mathrm{out}|\hat{\chi}(\eta_1)(i\omega)\hat{\chi}(\eta_0)|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} \ . \tag{21}$$

Hence the choice of initial state is encoded in the propagator as a boundary condition at η_0:

$$\partial_{\eta_2} iG_F(\eta_1,\eta_2)\big|_{\eta_2=\eta_0} = (i\omega) iG_F(\eta_1,\eta_0) \quad \text{for } \eta_1 > \eta_0 \ . \tag{22}$$

Inserting (20) into (22), we find that

$$u(\eta_1)u'^*(\eta_0) - \gamma u(\eta_1)u'(\eta_0) = i\omega\left(u(\eta_1)u^*(\eta_0) - \gamma u(\eta_1)u(\eta_0)\right) \ , \tag{23}$$

so that

$$\gamma = \frac{\omega + i\partial \ln u^*(\eta_0)}{\omega + i\partial \ln u(\eta_0)} \cdot \frac{u^*(\eta_0)}{u(\eta_0)} \ . \tag{24}$$

We can simplify this expression by defining[2]

$$\bar{\omega} \equiv i\partial_\eta \ln u(\eta)\big|_{\eta=\eta_0} \ . \tag{25}$$

[2]Throughout this paper overbars indicate quantities associated with the reflected problem we will introduce in section 4. They do not indicate complex conjugation, which we denote with a *.

The amount of pair creation is determined by $|\gamma|^2$, which is given by

$$|\gamma|^2 = \left|\frac{\omega - \bar{\omega}^*}{\omega + \bar{\omega}}\right|^2 \ . \tag{26}$$

This is the equation we are after. It tells us that given ω and $\bar{\omega}$ one knows the amount of pair production. Since ω parameterizes our choice of initial state, the only thing one has to compute is $\bar{\omega}$. This one obtains from (25) by differentiating the out-modes at the point η_0 where the initial conditions are specified. Since one takes a logarithmic derivative, there is no need to normalize the out-modes. More importantly, there is no need to determine any in-modes: particle production can be evaluated without computing Bogolubov coefficients. We present several examples of this economical formalism in the appendix.

4 First-Quantized Formalism

Our derivation so far is self-contained and sufficient for calculating pair production. However we do not yet have a first-quantized formalism, since to compute $\bar{\omega}$ from (25) we still need the late-time positive-frequency modes. In this section we show how to obtain $\bar{\omega}$ without using mode solutions, by considering the Feynman propagator in an auxiliary spacetime. This will lead us to a fully first-quantized language for discussing particle production.

One might imagine that the Green's function (18) could be expressed in first-quantized language as

$$iG_F(\eta_1, \eta_2) = \langle \eta_1 | \frac{i}{-D^2 + i\epsilon} | \eta_2 \rangle = \int_0^\infty ds\, e^{-\epsilon s} \int_{\eta(0)=\eta_2, \eta(s)=\eta_1} \mathcal{D}\eta(\cdot)\ e^{iS[\eta]} \ . \tag{27}$$

Here $D^2(\eta) = \partial_\eta^2 + V(\eta)$ is the differential (Schrödinger) operator that acts on the modes in (5). We have introduced an $i\epsilon$ prescription to make the Green's function well-defined, and given a path integral representation involving a Schwinger proper-time parameter s and a worldline action $S[\eta] = \int dt \left(-\frac{1}{4}\dot{\eta}^2 - V(\eta)\right)$. The basic difficulty with this representation is that, although the $i\epsilon$ prescription picks out definite in- and out-states, in a general curved background it is not clear exactly what those states are (p. 76 in [1]).

This difficulty can be overcome with the following trick. Consider a separate, auxiliary problem for which we reflect the spacetime symmetrically

about η_0. We then have a new operator, $\bar{D}^2(\eta)$, defined by

$$\bar{D}^2(\eta) \equiv \begin{cases} D^2(\eta) & \eta \geq \eta_0 \\ D^2(2\eta_0 - \eta) & \eta \leq \eta_0 \ . \end{cases} \tag{28}$$

The corresponding potential $\bar{V}(\eta)$ is the reflection of V about η_0. We define the Green's function in this reflected problem with an $i\epsilon$ prescription, so that it can be computed in first-quantized terms.

$$\begin{aligned} i\bar{G}_F(\eta_1,\eta_2) &= \langle \eta_1 | \frac{i}{-\bar{D}^2 + i\epsilon} | \eta_2 \rangle = \int_0^\infty ds\, e^{-\epsilon s} \int_{\eta(0)=\eta_2, \eta(s)=\eta_1} \mathcal{D}\eta(\cdot)\ e^{i\bar{S}[\eta]} \\ \bar{S}[\eta] &= \int_0^s dt \left(-\frac{1}{4} \left(\frac{d\eta}{dt} \right)^2 - \bar{V}(\eta) \right) \ . \end{aligned} \tag{29}$$

Notice that, as we are throwing away the part of the potential before η_0, any early-time singularity of the potential is irrelevant. The reflected potential may have a cusp at η_0 but this causes no problems. The advantage of reflecting the potential is that the spacetime is now asymptotically adiabatic in the past as well as in the future, so the $i\epsilon$ prescription automatically selects the preferred in- and out-vacua of the reflected problem. That is, as $\eta_1, \eta_2 \to \pm\infty$ the Green's function obeys the adiabatic boundary conditions considered in sect. 2.1. For example[3]

$$\partial_{\eta_2} \log i\bar{G}_F(\eta_1,\eta_2) \simeq i\sqrt{\bar{V}(\eta_2)} \quad \text{as } \eta_2 \to -\infty. \tag{31}$$

Suppose we are given the Feynman propagator $i\bar{G}_F$ in the reflected problem. It turns out that $\bar{\omega}$ can be extracted from this auxiliary propagator. To see this, imagine we knew the modes $\bar{u}(\eta)$ that satisfy the reflected differential equation $\bar{D}^2\bar{u}(\eta) = 0$. This is just a Schrödinger equation in a symmetric potential; a solution can be written in terms of the original unreflected modes as

$$\bar{u}(\eta) = \begin{cases} c_1 u^*(\eta) + c_2 u(\eta) & \eta \geq \eta_0 \\ u^*(2\eta_0 - \eta) & \eta \leq \eta_0 \ . \end{cases} \tag{32}$$

[3] The proof is as follows. The Feynman propagator $i\bar{G}_F(\eta_1,\eta_2) = \langle \eta_1 | \frac{i}{-\bar{D}^2 + i\epsilon} | \eta_2 \rangle$. Insert a complete set of energy eigenstates $\bar{D}^2|\omega\rangle = E(\omega)|\omega\rangle$ and let $\psi_\omega(\eta) = \langle \eta | \omega \rangle$. Adiabaticity implies that for $\eta \approx \eta_2$ we can take $\psi_\omega(\eta) \approx \exp(-i\omega\eta)$ with $E(\omega) \approx -\omega^2 + \bar{V}(\eta_2)$. Then

$$i\bar{G}_F(\eta_1,\eta_2) \approx \int_{-\infty}^{\infty} d\omega \rho(\omega) \frac{i}{\omega^2 - \bar{V}(\eta_2) + i\epsilon} \psi_\omega(\eta_1) e^{i\omega\eta_2} \tag{30}$$

where $\rho(\omega)$ is the density of states. The usual contour deformation picks up the pole at $\omega = \sqrt{\bar{V}(\eta_2)} - i\epsilon$, and (31) follows. Our approximations become exact as $\eta_2 \to -\infty$.

Recall that u is assumed to be positive frequency at late times, so $\bar{u}$ is a positive frequency mode as $\eta \to -\infty$. The coefficients c_1 and c_2 are fixed by requiring that $\bar{u}$ and its first derivative are continuous at η_0. We will not need explicit expressions for c_1 and c_2; indeed the whole point is that we won't need to know the modes at all.

The Feynman propagator in the auxiliary problem can be expressed in terms of the reflected modes:

$$\begin{aligned} i\bar{G}_F(\eta_1,\eta_2) &= \bar{u}(\eta_1)\bar{u}^*(\eta_2)\theta(\eta_{12}) + \bar{u}(\eta_2)\bar{u}^*(\eta_1)\theta(\eta_{21}) + \lambda_1\bar{u}(\eta_1)\bar{u}(\eta_2) \\ &\quad +\lambda_2[\bar{u}(\eta_1)\bar{u}^*(\eta_2) + \bar{u}^*(\eta_1)\bar{u}(\eta_2)] + \lambda_3\bar{u}^*(\eta_1)\bar{u}^*(\eta_2)\ , \end{aligned} \tag{33}$$

where the constants λ_i are coefficients of possible homogeneous terms. Every Feynman propagator between arbitrary in- and out-states can be written in this manner. We fix the λ_i by imposing boundary conditions corresponding to the initial and final states of the reflected problem, as determined by the $i\epsilon$ prescription. From (31) it follows that $\lambda_1 = \lambda_2 = 0$. Imposing the analogous equation at late times leads to an expression for λ_3, but the precise form is unimportant. Having set λ_1 and λ_2 to zero, we take the derivative of the reflected Green's function at η_0, and find

$$\partial_{\eta_2} i\bar{G}_F(\eta_1,\eta_2)\Big|_{\eta_1>\eta_2=\eta_0} = (\partial_\eta \log \bar{u}^*)\,|_{\eta=\eta_0}\ i\bar{G}_F(\eta_1,\eta_0)\ . \tag{34}$$

But

$$\partial_\eta \log \bar{u}^*|_{\eta=\eta_0} = -\partial_\eta \log u|_{\eta=\eta_0} = i\bar{\omega} \tag{35}$$

where we have used (32) and (25). So finally we arrive at

$$i\bar{\omega} = \partial_{\eta_2} \ln i\bar{G}_F(\eta_1,\eta_2)\Big|_{\eta_1>\eta_2=\eta_0}\ . \tag{36}$$

We see that we can determine $\bar{\omega}$ using only the propagator in the auxiliary reflected problem, without making any reference to mode functions. Representing $i\bar{G}_F$ as a particle path integral, this provides a fully first-quantized formalism for calculating cosmological pair production.

4.1 Path integrals and image charges

We conclude by showing how to represent the original Green's function iG_F in first-quantized language. Naively the propagator has the path integral representation (27); the difficulty is in understanding how to implement the correct boundary conditions at η_0.

To overcome this difficulty we express the original Green's function using the method of images, as

$$iG_F(\eta_1,\eta_2) = i\bar{G}_F(\eta_1,\eta_2) + qi\bar{G}_F(\eta_1, 2\eta_0 - \eta_2) \ , \tag{37}$$

where q is an image charge that we will determine shortly. For $\eta_1, \eta_2 > \eta_0$ note that iG_F is indeed a Green's function for the operator $D^2(\eta)$. It remains to implement the boundary condition at η_0 by choosing q appropriately. From (22) we require

$$\partial_{\eta_2} iG_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} = (i\omega) iG_F(\eta_1,\eta_0) \tag{38}$$

while from (37) we have

$$iG_F(\eta_1,\eta_0) = (1+q) i\bar{G}_F(\eta_1,\eta_0) \tag{39}$$

and

$$\partial_{\eta_2} i\, G_F(\eta_1,\eta_2)\big|_{\eta_1>\eta_2=\eta_0} = (1-q)\partial_{\eta_2} i\, \bar{G}_F(\eta_1,\eta_2)\Big|_{\eta_1>\eta_2=\eta_0} \ . \tag{40}$$

Using (36), we find that the image charge is given by

$$q = \frac{\bar{\omega} - \omega}{\bar{\omega} + \omega} \ . \tag{41}$$

The method of images also provides a way of representing the original propagator in terms of a particle path integral. Use (37) to represent iG_F as a sum of two particle path integrals in the auxiliary reflected potential. In each path integral fold the particle paths across $\eta = \eta_0$, to obtain particle worldlines that are restricted to satisfy $\eta(s) \geq \eta_0$. Note that this folding leaves the worldline action invariant. By adding the two path integrals one can represent iG_F as a single path integral, just as in (27), but where the particle paths are restricted to satisfy $\eta(s) \geq \eta_0$, and where the boundary conditions at $\eta = \eta_0$ are enforced by weighting paths according to the rule

$$\begin{array}{ll} e^{iS[\eta]} & \text{for paths that touch } \eta_0 \text{ an even number of times} \\ qe^{iS[\eta]} & \text{for paths that touch } \eta_0 \text{ an odd number of times} \end{array} \tag{42}$$

Similar constructions to enforce boundary conditions on the path integral are discussed in [16, 17].

5 Conclusions

To summarize: given a cosmological background and a set of initial conditions specified by the parameters ω_k, our recipe is to first compute the Green's function in the reflected potential $i\bar{G}_F$. This determines the quantities $\bar{\omega}_k$ and hence the ratio of Bogolubov coefficients $|\gamma_k|$. This is all that is needed for particle production, since $|\gamma_k|$ controls the amount of particle production. Since $i\bar{G}_F$ can be computed from a particle path integral, we have achieved our goal of finding a first-quantized formalism for computing cosmological particle production.

Our approach has nice conceptual advantages over conventional Bogolubov techniques. The formation of a pair of particles from the vacuum intuitively suggests a U-shaped particle worldline, with the two endpoints of the U marking the two particle positions at late times, and with the bottom of the U marking (heuristically) the time of pair creation. Our approach makes this intuition precise, by giving a prescription (42) for obtaining the appropriate Feynman propagator iG_F from an integral over particle paths. Moreover, in our approach the generalization to string theory is immediate: replace "particle worldlines" with "string worldsheets".

Acknowledgements

We are very grateful to Nori Iizuka for collaboration in the early stages of this project. We thank Brian Greene, Gilad Lifschytz, and Govindan Rajesh for valuable discussions. DK and MP are grateful to the organizers of the Amsterdam and Aspen workshops where part of this work was completed, and DK is grateful to the Rutgers theory group for its hospitality. AH is supported by DOE grant DE-FG02-92ER40699, DK is supported by DOE grant DE-FG02-92ER40699 and by US–Israel Bi-national Science Foundation grant #2000359, and MP is supported in part by DOE grant DF-FCO2-94ER40818.

6 Appendix: Examples

In this appendix, we demonstrate how the formalism of section 3 is applied by calculating pair production in various cosmological backgrounds. In practice, calculating particle production boils down to determining the late-time

positive-frequency modes and taking their derivative at the point η_0 at which the initial conditions are imposed.

6.1 The Milne universe

The line element for a two-dimensional Milne universe is

$$ds^2 = -dt^2 + s^2 t^2 dx^2 = e^{2s\eta}\left(-d\eta^2 + dx^2\right) \;, \tag{43}$$

where $-\infty < \eta, x < \infty$. This space is simply the upper quadrant of Minkowski space. Consider a massive scalar field propagating in this geometry. The wave equation is

$$\left(\partial_\eta^2 + k^2 + m^2 e^{2s\eta}\right) u_k(\eta) = 0 \;. \tag{44}$$

At early times the solutions are just plane waves so the natural choice for the in-vacuum is to set $\omega = k$. We will adopt this choice, which corresponds to the conformal vacuum [1], although we could easily consider different initial conditions. At late times the positive-frequency modes are Hankel functions:

$$u_k(\eta) = N H_\nu^{(2)}\left((m/s)e^{s\eta}\right) \;, \tag{45}$$

where $\nu = ik/s$ and N is a normalization constant that we will not need to determine. Suppose we specify initial conditions in the infinite far past, $\eta_0 \to -\infty$. Then we need the behavior of the out-mode as $\eta \to -\infty$. The Hankel function is proportional to $J_{-\nu}(x) - e^{-\pi k/s} J_\nu(x)$ and, as $x \to 0$,

$$J_\nu(x) \approx (x/2)^\nu / \Gamma(1+\nu) = A e^{ik\eta} \;, \tag{46}$$

where A is an η-independent constant. Particle production is controlled by

$$\bar{\omega} = i\partial_\eta \ln H_\nu^{(2)}(0) = k\frac{A^* e^{-ik\eta} + A e^{-\pi k/s} e^{ik\eta}}{A^* e^{-ik\eta} - A e^{-\pi k/s} e^{ik\eta}} \;. \tag{47}$$

From this we obtain γ_k:

$$|\gamma_k| = \left|\frac{k - \bar{\omega}^*}{k + \bar{\omega}}\right| = e^{-\pi k/s} \;. \tag{48}$$

The occupation number of a mode with momentum k is, using (17),

$$\langle \mathrm{in}|\hat{N}_k|\mathrm{in}\rangle = \frac{1}{e^{2\pi k/s} - 1} \;. \tag{49}$$

This indicates that modes in the Milne universe are, up to mass corrections, populated according to a Planck spectrum with temperature $T = s/2\pi$. This is in accord with the standard result that the conformal vacuum appears thermal to a comoving observer [1, 18, 19].

6.2 A doubly-asymptotically-static universe

Consider a two-dimensional universe with scale factor

$$a^2(\eta) = A + B \tanh \rho\eta \; , \tag{50}$$

where A, B, ρ are positive constants. The wave equation is

$$\left(\partial_\eta^2 + k^2 + m^2 \left(A + B \tanh \rho\eta\right)\right) u_k(\eta) = 0 \; . \tag{51}$$

This cosmology has no big bang; it is asymptotically static in the distant future as well as in the distant past. As $\eta \to \pm\infty$ the mode solutions behave like plane waves, albeit with different frequencies:

$$\omega_{\rm out} = (k^2 + m^2(A+B))^{1/2} \qquad \omega_{\rm in} = (k^2 + m^2(A-B))^{1/2} \; . \tag{52}$$

It is useful to define $\omega_\pm = (\omega_{\rm out} \pm \omega_{\rm in})/2$. The mode function with positive frequency at late times is

$$u_k(\eta) = N e^{-i\omega_+\eta} (\cosh \rho\eta)^{-i\omega_-/\rho} F(a, b; c; (1/2)(1 - \tanh \rho\eta)) \; , \tag{53}$$

where F is a hypergeometric function with

$$a = 1 + i\omega_-/\rho \qquad b = i\omega_-/\rho \qquad c = 1 + i\omega_{\rm out}/\rho \; . \tag{54}$$

In the conventional approach [20] one would have to determine the modes that are positive-frequency at early times; these are just as messy as the late-time modes. One then has to find the linear transformation that relates the two sets of modes. But in our approach it suffices to note that the positive-frequency modes behave as plane waves at early times. Thus a natural choice is to set $\omega = \omega_{\rm in}$. To compute the particle production we need the logarithmic derivative of the out-modes evaluated in the far past. Defining $x = (1/2)(1 - \tanh \rho\eta)$, we expand F around $x = 1$:

$$F(a, b; c; x) \approx (1-x)^{c-a-b} A + ... + B + ... \; , \tag{55}$$

where the dots indicate terms that are unimportant as $x \to 1$. Here A and B are

$$A = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(a)\Gamma(b)} \qquad B = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \; . \tag{56}$$

Then as $\eta \to -\infty$ or $x \to 1$, we have

$$\bar{\omega} = \omega_{\rm in} \left(1 - \frac{2A(1-x)^{iw_{\rm in}/\rho}}{(1-x)^{iw_{\rm in}/\rho} A + B}\right)\Bigg|_{x=1} \; . \tag{57}$$

Some simple algebra yields

$$|\gamma| = |A/B| = \left| \frac{\Gamma(i\omega_+/\rho)\Gamma(1 + i\omega_+/\rho)}{\Gamma(i\omega_-/\rho)\Gamma(1 + i\omega_-/\rho)} \right| , \tag{58}$$

which agrees precisely with the result obtained by conventional methods [20, 1].

6.3 Power-law FRW

In the two preceding examples the mode solutions reduce to plane waves at early times. A natural choice for the in-vacuum was therefore one in which the annihilation operator $\hat{b}_k$ is paired with $\exp(-i\omega_{\rm in}\eta)$ so that $\omega = \omega_{\rm in}$. In a general cosmological background, however, there is no preferred vacuum state at early times, and ω is just a parameter that characterizes the initial state. For example consider a 3+1-dimensional Robertson-Walker spacetime with a power-law scale factor, $a(t) \sim t^c$. We consider $0 < c < 1$ so the expansion is decelerating. The line element reads

$$ds^2 = -dt^2 + a^2(t)d\Sigma_3^2 = C^2 \eta^{\frac{2c}{1-c}}(-d\eta^2 + d\Sigma_3^2) , \tag{59}$$

where $0 < \eta < \infty$ and C is an unimportant constant. A massless minimally-coupled real scalar field $\phi(x)$ can be conveniently written as $\phi(x) = \chi(x)/a(\eta)$. The mode equation is

$$\left(\partial_\eta^2 + k^2 - \frac{\nu^2 - 1/4}{\eta^2} \right) \chi_k(\eta) = 0 , \quad \nu = \frac{3c - 1}{2(1 - c)} . \tag{60}$$

If $\nu^2 \neq 1/4$ the solutions do not resemble plane waves at early times and there is no natural choice for ω. Let us pick some arbitrary time η_0 at which to fix our choice of initial state. Define $\hat{\chi}_k(\eta_0) = \hat{\chi}_0$ and $(\partial_\eta \hat{\chi}_k)(\eta_0) = \hat{\pi}_0$. As before, define an operator

$$\hat{b}_k = N_k \left(\hat{\chi}_0 + i \frac{\hat{\pi}_0}{\omega_k} \right) , \tag{61}$$

and set $\hat{b}_k|\text{in}\rangle = 0$. This implies that

$$\hat{\pi}_0|\text{in}\rangle = i\omega_k \hat{\chi}_0|\text{in}\rangle . \tag{62}$$

The out-vacuum is unique, since the field evolves adiabatically at late times. The positive-frequency modes at late times are

$$u_k(\eta) = N\sqrt{\eta} H_\nu^{(2)}(k\eta) \ , \tag{63}$$

so that

$$\bar{\omega} = i \, \partial_\eta \ln \sqrt{\eta} H_\nu^{(2)}(k\eta)\Big|_{\eta=\eta_0} \ . \tag{64}$$

To see what happens as $\eta_0 \to 0$, expand $u_k(\eta)$ around $\eta = 0$, keeping the lowest order real and imaginary terms:

$$\begin{aligned}
\bar{\omega} &\approx i\partial_\eta \ln \left[A \left(\frac{k\eta}{2} \right)^{\nu+1/2} + iB \left(\frac{k\eta}{2} \right)^{1/2-\nu} \right] \Bigg|_{\eta=\eta_0} \\
&= i\partial_\eta \ln \left[\left(1 - i\frac{A}{B} \left(\frac{k\eta}{2} \right)^{2\nu} \right) \left(iB \left(\frac{k\eta}{2} \right)^{1/2-\nu} \right) \right] \Bigg|_{\eta=\eta_0} \\
&\approx \frac{k\nu\Gamma(1-\nu)}{\Gamma(1+\nu)} \sin(\nu\pi) \left(\frac{k\eta_0}{2} \right)^{2\nu-1} + \frac{2\nu-1}{2i\eta_0} \ ,
\end{aligned} \tag{65}$$

where $A = \sqrt{\frac{2}{k}} \frac{1}{\Gamma(1+\nu)}$, $B = \sqrt{\frac{2}{k}} \frac{\csc(\nu\pi)}{\Gamma(1-\nu)}$, and in going from the second to the third line we expanded the log about 1. Particle production is controlled by

$$|\gamma| \approx \left| \frac{\omega - \frac{k\nu\Gamma(1-\nu)}{\Gamma(1+\nu)} \sin(\nu\pi) \left(\frac{k\eta_0}{2} \right)^{2\nu-1} + i\frac{(\frac{1}{2}-\nu)}{\eta_0}}{\omega + \frac{k\nu\Gamma(1-\nu)}{\Gamma(1+\nu)} \sin(\nu\pi) \left(\frac{k\eta_0}{2} \right)^{2\nu-1} + i\frac{(\frac{1}{2}-\nu)}{\eta_0}} \right| \ . \tag{66}$$

This approaches unity for $\nu^2 \neq 1/4$, since the imaginary term dominates as $\eta_0 \to 0$. Hence the pair production diverges, a result which perhaps is merely a sign that the initial condition should not be applied strictly at $\eta_0 = 0$ where the geometry is singular. However, for $\nu = \pm 1/2$, which corresponds to $c = 0$ (a static universe) or $c = 1/2$ (a radiation-dominated universe), $|\gamma|$ simplifies greatly to

$$|\gamma| = \left| \frac{k-\omega}{k+\omega} \right| . \tag{67}$$

This result follows from the fact that for $\nu^2 = 1/4$ the Hankel function reduces to a plane wave with energy k. If one sets $\omega = k$ then $|\text{in}\rangle$ is identical to $|\text{out}\rangle$, so there should be no particle production, and indeed for this value of ω we see that γ vanishes.

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